Optimal Strapdown Attitude Integration Algorithms

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The paper examines various algorithms for integrating the noncommutivity rate equation arising in the implementation of the attitude reference function of a strapdown inertial navigation system, and derives a class of optimized algorithms for performing this integration. The accuracy characteristics associated with each of the algorithms, when the system is exposed to pure coning motion, as well as to a generalized vibrational environment, are defined. The class of optimized algorithms is shown to minimize the mean error in the correction provided by the algorithm in a generalized vibrational environment.

Introduction

AN important concern in the design of strapdown inertial navigation systems is the propagation of attitude error in a highly dynamic angular motion environment, in which the angular motions about the individual axes of the inertial sensor assembly and the relative phasings of these motions may be such that a mean angular drift rate will result about one or more axes if not properly accounted for in the system attitude reference computations. A generalized treatment of the attitude reference problem, as given by Bortz[1] and Jordan,[2] shows that the high-frequency angular motions of concern can be accounted for by separately integrating a vector differential equation (variously referred to as the "coning" or "noncommutivity rate" equation) at a substantially higher rate than that associated with the basic attitude matrix update, using a compatible high-frequency stream of incremental angle data from the strapdown gyros. Then, when the solution thus obtained is added to the integral of the angular rate vector over a given attitude update interval, a total attitude increment results that may be processed by the attitude matrix update algorithm over this longer interval, thereby leading to a very efficient implementation of the attitude reference solution since only a relatively small part of the computations needs to be carried out at the highest iteration rate. Given this dual-frequency structure for the attitude update scheme, a separate issue arises concerning the best choice of algorithm for integrating the noncommutivity rate equation. A recent advancement in this area is due to Miller,[3] whose results lead to a higher-order integration algorithm that produces an optimal result in a pure coning environment. The present paper examines the coning algorithm design problem and defines the accuracies associated with various options for mechanizing the coning correction, including a generalized form of the Miller algorithm and, additionally, shows that the algorithm optimization procedure employed by Miller carries over to a generalized vibrational environment as well.

Attitude Update Equation

The following equation must be solved numerically to maintain knowledge of the attitude of the system:

$$\dot{C} = C\{\omega\} \tag{1}$$

where C is the transformation matrix from the rotating frame to the fixed frame, ω is the angular rate vector of the rotating frame relative to the fixed frame, and $\{\cdot\}$ denotes a skew-symmetric matrix formed from the components of the enclosed vector.

The solution of the continuous attitude equation defined by Eq. (1) is accomplished by a recursive attitude update equation, which approximates the exact solution by a series expansion of the form

$$C_n = C_{n-1}(I + \{\Delta\theta_n\} + ½\{\Delta\theta_n\}^2 + \cdots) \tag{2}$$

where $\{\Delta\theta\}$ is the incremental rotation matrix of the rotating frame relative to the fixed frame, computed according to

$$\Delta\theta_n = \int_{t_{n-1}}^{t_n} \omega \, dt + \Delta\theta_c(n) \tag{3}$$

in which t_n denotes the time at the end of the nth major computational interval, and $\Delta\theta_c(n)$ is the integral of the noncommutivity rate vector, usually approximated as in Ref. (3) by

$$\Delta\theta_c(n) = ½\int_{t_{n-1}}^{t_n} \theta(t, t_{n-1}) \times \omega \, dt \tag{4}$$

with $\theta(t, t_{n-1})$ denoting

$$\theta(t, t_{n-1}) = \int_{t_{n-1}}^{t} \omega \, dt \tag{5}$$

The term $\Delta\theta_c$, referred to as the "coning correction," is computed by sampling the gyro outputs at a substantially higher rate than the basic attitude matrix update frequency and, by so doing, the accuracy of the attitude reference solution is preserved for relatively high-frequency angular rate inputs, such as might arise in a vibrational environment.

Coning Algorithms and Accuracies

Nine algorithms representing different realizations of the coning correction equation defined in Eq. (4), each directly utilizing the incremental-angle outputs from the three strapdown gyros, are defined as

Algorithm A

$$\Delta\theta_c(n) = \frac{1}{2}\sum_{m=2}^{M} \theta_{m-1} \times \Delta\theta_m$$

Algorithm B

$$\Delta\theta_c(n) = \frac{1}{2}\sum_{m=2}^{M} \theta_{m-1} \times \Delta\theta_m + \frac{1}{12}\sum_{m=1}^{M} \Delta\theta_{m-1} \times \Delta\theta_m$$

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Algorithm C

$$\Delta\theta_c(n)=\frac{1}{2}\sum_{m=2}^{M}\theta_{m-1}\times\Delta\theta_m+\frac{2}{3}\sum_{m=1}^{M}\Delta\theta_m(1)\times\Delta\theta_m(2)$$

Algorithms D and E

$$\Delta\theta_c(n)=\frac{1}{2}\sum_{m=2}^{M}\theta_{m=1}\times\Delta\theta_m+A\sum_{m=2}^{M}\Delta\theta_m(1)\times\Delta\theta_m(3)$$

$$+B\sum_{m=1}^{M}\Delta\theta_m(2)\times[\Delta\theta_m(3)-\Delta\theta_m(1)]$$

where

$A=33/80,\quad B=57/80\quad$ for Algorithm D

$A=9/20,\quad B=27/40\quad$ for Algorithm E

Algorithm E′

$$\Delta\theta_c(n)=\frac{1}{2}\sum_{m=2}^{M}\theta_{m-1}\times\Delta\theta_m$$

$$+A\sum_{m=1}^{M}\left[[\Delta\theta_m(1)-\Delta\theta_m(3)]\times[\Delta\theta_m(3)+\frac{3}{2}\Delta\theta_m(2)]\right]$$

where

$A=9/20$

Algorithm F

$$\Delta\theta_c(n)=\frac{1}{2}\sum_{m=2}^{M}\theta_{m-1}\times\Delta\theta_m$$

$$+A\sum_{m=1}^{M}[\Delta\theta_m(1)+3\Delta\theta_m(2)]\times\Delta\theta_m(3)$$

where

$A=9/20$

Algorithm G

$$\Delta\theta_c(n)=\frac{1}{2}\sum_{m=2}^{M}\theta_{m-1}\times\Delta\theta_m$$

$$+A\sum_{m=1}^{M-1}[\Delta\theta_m(1)+3\Delta\theta_m(2)]\times\Delta\theta_m(3)$$

$$+B[\Delta\theta_{M-1}(2)-7\Delta\theta_{M-1}(3)]\times\Delta\theta_M(1)$$

where

$A=9/20,\quad B=-1/60$

Algorithm H

$$\Delta\theta_c(n)=\frac{1}{2}\sum_{m=2}^{M}\theta_{m-1}\times\Delta\theta_m$$

$$+A\sum_{m=1}^{M-1}[\Delta\theta_m(1)+3\Delta\theta_m(2)]\times\Delta\theta_m(3)$$

$$+B[\Delta\theta_{M-1}(3)-22\Delta\theta_M(1)]\times\Delta\theta_M(2)$$

where

$A=9/20,\quad B=-1/30$

and in which the following definitions apply:

$\Delta\theta_m=$ gyro incremental angle output over mth minor interval of the nth major interval

$\theta_m=$ gyro incremental angle output from beginning of nth major interval to end of mth minor interval

$\Delta\theta_m(i)=$ gyro incremental angle output over ith subminor interval of mth minor interval

$\Delta\theta_c(n)=$ coning correction over nth major computational interval

$M=$ number of minor intervals in major computational interval

The coning algorithms assume that each major attitude interval is divided into a number of minor intervals, each in turn being divided into a number of subminor intervals over which the gyro incremental angle data is available, as depicted in Fig. 1. This constitutes the most general structure possible for the algorithms, since no restriction is placed on the number of sensor data intervals contained in a major attitude update interval (as is the case in Ref. 3, for example).

Algorithm A is based on making the approximation over each interval that θ is constant, allowing $\theta\times$ to be removed from the integration in Eq. (4), which then allows the resultant expression to be analytically integrated. This constitutes the simplest algorithm considered for integrating Eq. (4).

Algorithm B improves upon algorithm A by correcting for the assumption of constant θ over each minor interval. The correction term is based on the assumption that the angular rate vector ω varies linearly with time over the duration of the minor interval, and is implemented using the incremental gyro outputs over the previous as well as the present minor interval. This algorithm is the generalized form of a coning algorithm given in Refs. 4 and 5.

Algorithm C is based on the same assumption implicit in Algorithm B. The difference is that the correction term is implemented using the gyro incremental outputs over two subminor intervals of each minor interval. A specialized form of this algorithm, which assumes two sensor data intervals per major attitude update interval, is given in Ref. 2.

Algorithms D and E are based on Miller's work,[3] but represent the generalized form of his result, insofar as the restriction of three sensor data intervals over the major attitude update interval is removed. The algorithms use Millers' result to compute the correction term and, as in Ref. 3, assume that the angular rate vector varies as a quadratic in time over the duration of the minor interval. The correction term is implemented using the gyro incremental angle outputs over three subminor intervals spanning the minor interval. The coefficients A and B have two sets of values, the second set (algorithm E) resulting in a minimization of algorithm error in a pure coning environment.

Algorithm E′ is a rearranged version of algorithm E; it has the same accuracy characteristics as algorithm E but is computationally more efficient.

Algorithm F is a variant of algorithm E with the same accuracy characteristics but a reduced computational burden associated with it. Algorithms G and H are also variants of algorithm E, and are intended to be used when the major computational interval cannot be divided into a number of sensor data intervals that is an integer multiple of three, as required by algorithms E and F.

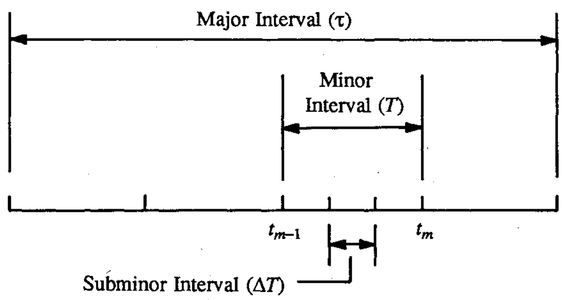

Fig. 1 Intervals associated with computation of coning correction.

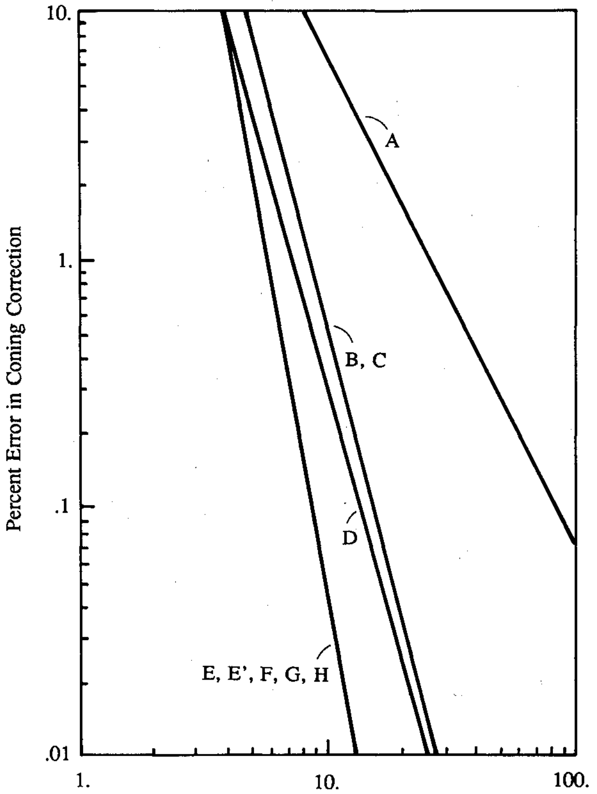

Fig. 2 Accuracy of coning correction provided by various algorithms.

The accuracy associated with each algorithm in a pure coning environment is shown in Fig. 2. The accuracy curves may be ascertained using either of two techniques. The first is based on implementing each algorithm by means of a digital computer simulation, and assessing the coning correction provided by the algorithm using a stream of synthetic sensor data derived for an assumed pure coning environment; then, comparing with an analytical representation of the exact coning correction that exists for the assumed angular rate, the algorithm error is found as the difference between the two corrections. The second technique employs a closed-form representation of the algorithm error in a pure coning environment, some examples of this type of algorithm error analysis being given in the subsequent development. The algorithm accuracy results of Fig. 2 are given in a normalized fashion, with the percent error in the coning correction plotted against the ratio of sensor data input frequency to coning frequency. The clear superiority of the optimized algorithms (E, E′, F, G, and H) is noteworthy.

Relationships in a Pure Coning Environment

As a preliminary to the algorithm derivation and optimization methodology to follow, two useful relationships will be derived that are applicable in a pure coning environment. These relationships will serve as basic building blocks in the subsequent development.

The first relationship of interest defines the coning correction over a minor computational interval. The second defines the cross product of two incremental attitude vectors taken over different subminor intervals.

Assume that the body is undergoing pure coning motion, defined by the angular rate vector

$$\boldsymbol{\omega} = a\Omega \cos\Omega t \boldsymbol{I} + b\Omega \sin\Omega t \boldsymbol{J} \tag{6}$$

where

$\boldsymbol{\omega}$ = angular velocity vector with components expressed in the body frame
a, b = amplitudes of the angular oscillations in two orthogonal axes of the body
Ω = frequency associated with the angular oscillations
$\boldsymbol{I}, \boldsymbol{J}$ = unit vectors along the two body axes about which the oscillations are occurring

From the definition of the coning corrector, the value obtained for the coning correction over a minor computational interval (from t_{m-1} to t_m) may be determined in the following manner.

First, determine the vector $\boldsymbol{\theta}$ (t, t_{m-1}) from Eqs. (5) and (6) as

$$\boldsymbol{\theta}(t, t_{m-1}) = a(\sin\Omega t - \sin\Omega t_{m-1})\boldsymbol{I} - b(\cos\Omega t - \cos\Omega t_{m-1})\boldsymbol{J}$$

Then, taking the cross product of $\boldsymbol{\theta}$ (t, t_{m-1}) with $\boldsymbol{\omega}$, as defined by Eq. (6), gives

$$\boldsymbol{\theta}(t, t_{m-1}) \times \boldsymbol{\omega} = ab\Omega[1 - \cos\Omega(t - t_{m-1})]\boldsymbol{K}$$

which allows the coning correction over the mth minor computational interval, as defined by Eq. (4), to be determined as

$$\Delta\boldsymbol{\theta}_c = \frac{ab\Omega}{2}[t_m - t_{m-1} - \frac{1}{\Omega}\sin\Omega(t_m - t_{m-1})]\boldsymbol{K}$$

where $\Delta\boldsymbol{\theta}_c$ is used here and throughout to denote the coning correction over a minor computational interval. Since the time difference is equal to the duration of a minor interval T, we have that

$$\Delta\boldsymbol{\theta}_c = \frac{ab\Omega}{2}(T - \frac{1}{\Omega}\sin\Omega T)\boldsymbol{K} \tag{7}$$

The result given by Eq. (7) reveals the interesting property that the coning correction is constant over all minor intervals, regardless of the absolute time at which the interval begins, but depends only on the duration of the minor interval T.

The second basic relationship of interest involves the cross product of two incremental attitude vectors $\Delta\boldsymbol{\theta}_i \times \Delta\boldsymbol{\theta}_j$ taken over different subminor intervals. Consider first, for the coning environment defined by Eq. (6), the incremental attitude vector over a subminor interval of duration ΔT ending at t_k:

$$\Delta\boldsymbol{\theta}_k = \int_{t_{k-1}}^{t_{k-1}+\Delta T} \boldsymbol{\omega}\, \mathrm{d}t = \int_{t_{k-1}}^{t_{k-1}+\Delta T} \Omega(a\cos\Omega t\boldsymbol{I} + b\sin\Omega t\boldsymbol{J})\, \mathrm{d}t$$

which, after integration, trigonometric expansion, and consolidation of terms reduces to

$$\Delta\boldsymbol{\theta}_k = a(-\alpha\sin\Omega t_{k-1} + \beta\cos\Omega t_{k-1})\boldsymbol{I} + b(\alpha\cos\Omega t_{k-1} + \beta\sin\Omega t_{k-1})\boldsymbol{J}$$

where α and β are defined by

$$\alpha = 1 - \cos\Omega\Delta T, \qquad \beta = \sin\Omega\Delta T$$

The cross product $\Delta\boldsymbol{\theta}_i \times \Delta\boldsymbol{\theta}_j$ is evaluated as

$$\Delta\boldsymbol{\theta}_i \times \Delta\boldsymbol{\theta}_j = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{J} & \boldsymbol{K} \\ -\alpha\sin\Omega t_{i-1} + \beta\cos\Omega t_{i-1} & \alpha\cos\Omega t_{i-1} + \beta\sin\Omega t_{i-1} & 0 \\ -\alpha\sin\Omega t_{j-1} + \beta\cos\Omega t_{j-1} & \alpha\cos\Omega t_{j-1} + \beta\sin\Omega t_{j-1} & 0 \end{bmatrix}$$

Carrying out the cross product and combining terms leads to the result

$$\Delta\boldsymbol{\theta}_i \times \Delta\boldsymbol{\theta}_j = ab\{-(\alpha^2+\beta^2)\sin[(i-j)\Omega\Delta T]\}\boldsymbol{K}$$

Then, making use of the identity

$$\alpha^2+\beta^2 = 2(1-\cos\Omega\Delta T)$$

the desired result is obtained as

$$\Delta\boldsymbol{\theta}_i \times \Delta\boldsymbol{\theta}_j = ab\{2\sin[(j-i)\lambda] - \sin[(j-i+1)\lambda] - \sin[(j-i-1)\lambda]\}\mathbf{K} \tag{8}$$

where $\lambda = \Omega\Delta T$. Like the coning correction, the value of the cross product of two attitude increments is independent of absolute time, but depends only on the duration ΔT of the intervals and their spacing.

Derivation of Coning Algorithms D, E, E′, F, G, and H

The derivation and optimization of the class of higher-order algorithms to which algorithms D, E, F, G, and H belong are carried out in this section. All of the other algorithms defined previously may be derived in the same manner.

Derivation of Coning Algorithm D

Over the nth major computational interval, the coning correction is computed from Eq. (4), and this can be carried out by dividing the major computational interval into a number of minor intervals, each of which in turn can be divided into a number of subminor intervals. Therefore, it is possible to express the coning correction over a major computational interval as the sum of the contributions from the M minor intervals, each computed according to Eq. (4), as

$$\begin{aligned}\Delta\boldsymbol{\theta}_c(n) &= \sum_{m=1}^{M}\frac{1}{2}\int_{t_{m-1}}^{t_m}\boldsymbol{\theta}(t,\ t_0)\times\boldsymbol{\omega}\ \mathrm{d}t \\ &= \frac{1}{2}\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}[\boldsymbol{\theta}_{m-1}+\boldsymbol{\theta}(t,\ t_{m-1})]\times\boldsymbol{\omega}\ \mathrm{d}t \\ &= \frac{1}{2}\sum_{m=2}^{M}\boldsymbol{\theta}_{m-1}\times\Delta\boldsymbol{\theta}_m \\ &\quad +\frac{1}{2}\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}\boldsymbol{\theta}(t,\ t_{m-1})\times\boldsymbol{\omega}\ \mathrm{d}t\end{aligned} \tag{9}$$

in which the following definitions apply:

$$\boldsymbol{\theta}(t,\ t_{m-1}) = \int_{t_{m-1}}^{t_m}\boldsymbol{\omega}\ \mathrm{d}t$$

$$\Delta\boldsymbol{\theta}_m = \int_{t_{m-1}}^{t_m}\boldsymbol{\omega}\ \mathrm{d}t$$

$$\boldsymbol{\theta}_{m-1} = \boldsymbol{\theta}(t_{m-1},\ t_0) = \sum_{i=1}^{m-1}\Delta\boldsymbol{\theta}_i$$

and where t_0 denotes the time associated with the beginning of the particular major interval.

Assume that the angular rate vector $\boldsymbol{\omega}$ can be closely approximated over the three contiguous subminor intervals of the given minor interval by the second-order polynomial

$$\boldsymbol{\omega} = \boldsymbol{A} + \boldsymbol{B}(t-t_{m-1}) + \boldsymbol{C}(t-t_{m-1})^2 \tag{10}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are vector constants unique to the given interval consisting of three concatenated subminor intervals. The integral of $\boldsymbol{\omega}$ over the given interval is obtained by integrating Eq. (10), and results in

$$\boldsymbol{\theta}(t,\ t_{m-1}) = \boldsymbol{A}(t-t_{m-1}) + \tfrac{1}{2}\boldsymbol{B}(t-t_{m-1})^2 + \tfrac{1}{3}\boldsymbol{C}(t-t_{m-1})^3 \tag{11}$$

Then, from Eqs. (10) and (11), the coning correction over the mth minor interval is determined as

$$\begin{aligned}\Delta\boldsymbol{\theta}_c &= \frac{1}{2}\int_{t_{m-1}}^{t_{m-1}+3\Delta T}\boldsymbol{\theta}(t,\ t_{m-1})\times\boldsymbol{\omega}\ \mathrm{d}t \\ &= \left[\frac{9}{4}\boldsymbol{A}\times\boldsymbol{B}\Delta T^3 + \frac{27}{4}\boldsymbol{A}\times\boldsymbol{C}\Delta T^4 + \frac{81}{20}\boldsymbol{B}\times\boldsymbol{C}\Delta T^5\right]\end{aligned} \tag{12}$$

The coefficients $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ can be found by using the relationship between these coefficients and the incremental attitude vectors occurring over three contiguous subminor intervals. Therefore, define the incremental angle vector over the ith subminor interval of the mth minor interval by $\Delta\boldsymbol{\theta}_m(i)$; then, in terms of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, the following relationships are found:

$$\Delta\boldsymbol{\theta}_m(1) = \boldsymbol{A}\Delta T + \boldsymbol{B}\frac{\Delta T^2}{2} + \boldsymbol{C}\frac{\Delta T^3}{3} \tag{13a}$$

$$\Delta\boldsymbol{\theta}_m(2) = \boldsymbol{A}\Delta T + \frac{3}{2}\boldsymbol{B}\Delta T^2 + \frac{7}{3}\boldsymbol{C}\Delta T^3 \tag{13b}$$

$$\Delta\boldsymbol{\theta}_m(3) = \boldsymbol{A}\Delta T + \frac{5}{2}\mathbf{B}\Delta T^2 + \frac{19}{3}\boldsymbol{C}\Delta T^3 \tag{13c}$$

Solving Eqs. (13) for $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, gives

$$\boldsymbol{A} = \frac{1}{\Delta T}\left[\frac{11}{6}\Delta\boldsymbol{\theta}_m(1) - \frac{7}{6}\Delta\boldsymbol{\theta}_m(2) + \frac{1}{3}\Delta\boldsymbol{\theta}_m(3)\right] \tag{14a}$$

$$\boldsymbol{B} = \frac{1}{\Delta T^2}\left[-2\Delta\boldsymbol{\theta}_m(1) + 3\Delta\boldsymbol{\theta}_m(2) - \Delta\boldsymbol{\theta}_m(3)\right] \tag{14b}$$

$$\boldsymbol{C} = \frac{1}{\Delta T^3}\left[\frac{1}{2}\Delta\boldsymbol{\theta}_m(1) - \Delta\boldsymbol{\theta}_m(2) + \frac{1}{2}\Delta\boldsymbol{\theta}_m(3)\right] \tag{14c}$$

Substituting into Eq. (12) yields the coning correction over the mth minor computational interval as

$$\Delta\boldsymbol{\theta}_c = \frac{33}{80}\Delta\boldsymbol{\theta}_m(1)\times\Delta\boldsymbol{\theta}_m(3) + \frac{57}{80}\Delta\boldsymbol{\theta}_m(1)\times\Delta\boldsymbol{\theta}_m(2) + \frac{57}{80}\Delta\boldsymbol{\theta}_m(2)\times\Delta\boldsymbol{\theta}_m(3) \tag{15}$$

Then, substituting this result into Eq. (9) leads directly to algorithm D.

Derivation of Coning Algorithms E and E′

The type of algorithm defined by algorithm D is subject to optimization[3]; the optimization procedure implicitly assumes that a pure coning environment exists. When such is the case, it is a straightforward matter to determine the error in the correction provided by the coning algorithm by comparing the exact value obtained from Eq. (7) with that obtained by the algorithm itself, as determined from the form implicit in Eq. (15).

The exact coning correction over a minor computational interval T is

$$\Delta\boldsymbol{\theta}_c = \frac{ab}{2}(3\Omega\Delta T - \sin 3\Omega\Delta T)\boldsymbol{K} = \frac{ab}{2}(3\lambda - \sin 3\lambda)\boldsymbol{K} \tag{16}$$

Expanding into a series approximation yields the result

$$\Delta\boldsymbol{\theta}_c = \frac{ab}{2}\left[\frac{(3\lambda)^3}{6} - \frac{(3\lambda)^5}{120} + \frac{(3\lambda)^7}{5040} - \cdots\right]\boldsymbol{K} \tag{17}$$

The correction provided by algorithm D can be determined from the contributions of its individual terms, evaluated from Eq. (8), as

$$A\,\Delta\boldsymbol{\theta}_m(1)\times\Delta\boldsymbol{\theta}_m(3)=abA(2\sin 2\lambda-\sin\lambda-\sin 3\lambda)\boldsymbol{K} \tag{18}$$

$$B\,\Delta\boldsymbol{\theta}_m(1)\times\Delta\boldsymbol{\theta}_m(2)=abB(2\sin\lambda-\sin 2\lambda)\boldsymbol{K} \tag{19}$$

$$C\,\Delta\boldsymbol{\theta}_m(2)\times\Delta\boldsymbol{\theta}_m(3)=abC(2\sin\lambda-\sin 2\lambda)\boldsymbol{K} \tag{20}$$

Expanding each term in Eqs. (18-20) as a series approximation, and adding, yields the series expansion for the algorithmic coning correction $\Delta\tilde{\boldsymbol{\theta}}_c$ over the minor computational interval, as

$$\Delta\tilde{\boldsymbol{\theta}}_c=ab\left[A\left(2\lambda^3-\frac{3}{2}\lambda^5+\frac{161}{420}\lambda^7-\cdots\right)+\left(B+C\right)\left(\lambda^3-\frac{1}{4}\lambda^5+\frac{63}{2520}\lambda^7-\cdots\right)\right]\boldsymbol{K} \tag{21}$$

Differencing terms in Eqs. (17) and (21) involving the same power of λ provides two conditions that, if both satisfied, will cause the coefficient of the λ^3 and λ^5 terms to be zero. The two conditions are

$$\frac{9}{4}-2A-(B+C)=0 \tag{22}$$

and

$$-\frac{81}{80}+\frac{3}{2}A+\frac{B+C}{4}=0 \tag{23}$$

For the case where it is assumed that $C=B$, Eqs. (22) and (23) may be solved simultaneously to yield the optimal values for the coefficients A and B, which are found to be $A=9/20$ and $B=27/40$. This leads to algorithm E, a simplified form of which can be obtained by algebraic manipulation as follows. Combining the B and C terms and then noting that the cross product of $\Delta\boldsymbol{\theta}_m(3)$ with itself as zero, the A term can be added to the sum thus determined, giving the result

$$\Delta\boldsymbol{\theta}_c=\frac{9}{20}[\Delta\boldsymbol{\theta}_m(1)-\Delta\boldsymbol{\theta}_m(3)]\times\left[\Delta\boldsymbol{\theta}_m(3)+\frac{3}{2}\,\Delta\boldsymbol{\theta}_m(2)\right] \tag{24}$$

This leads to algorithm E′, which involves one fewer cross product than algorithm E.

Derivation of Coning Algorithm F

Suppose the constraint that B and C are equal is removed. Then, it is a straightforward matter to show from Eqs. (22) and (23) that the conditions that lead to the optimal choice of A and B are

$$2A+(B+C)=\frac{9}{4}$$

and

$$6A+(B+C)=\frac{81}{20}$$

which, in turn, lead to the optimal coefficient values of $A=9/20$ and $B+C=27/20$. The result shows that as long as B and C are chosen such that their sum equals the given value, the accuracy of the algorithm in the pure coning environment assumed is the same as that provided by algorithm E. It is clear, therefore, that either B or C can be chosen as zero and, by so doing, the algorithm is simplified. For the choice $B=0$, the value of C is 27/20, and algorithm F results. A choice of $C=0$ is also possible, and leads to a value of $B=27/20$. However, all properties of the resulting algorithm are the same. Therefore, this variant is not considered further.

Derivation of Coning Algorithms G and H

It may happen that, due to a constraint on the choice of frequencies that are available in implementing the attitude update computations, it is not possible to subdivide a major computational interval into an integer number of sensor data intervals that is a multiple of three, as required by algorithms E and F. An effective way to deal with this type of constraint is to use algorithm E or F over the set of subminor intervals that is divisible by three, and then compute the coning correction over the remainder of the major interval by means of special algorithms. Algorithms G and H serve this purpose.

Algorithms G and H are based on a second-order polynomial approximation for the angular rate vector [as defined by Eq. (10) over three contiguous subminor intervals]. The difference between these two algorithms and algorithms E and F is that the coning correction is not computed over the full set of three subminor intervals, but over only a portion of the total interval spanned by the polynomial. The derivation of two algorithms, which are variants of algorithm E, and which have the same optimality properties, follows.

Algorithm G

Consider the first special case leading to algorithm G. The situation assumed here is one where a single interval is left over after applying either algorithm E, E′, or F to a set of subminor intervals that is some multiple of three. The coning correction over this remaining interval is then computed using the sensor data over this interval together with that occurring over the previous minor interval.

The structure of algorithm G is taken to be the same as that for algorithm D, with the exception that the final minor interval consists of a single subminor interval, over which the coning correction is computed using sensor data occurring in this interval, and sensor data occurring over the last two subminor intervals of the previous minor interval. The structure of the algorithm is as follows:

$$\Delta\boldsymbol{\theta}_c=A\,\Delta\boldsymbol{\theta}_{m-1}(2)\times\Delta\boldsymbol{\theta}_m(1)+B\,\Delta\boldsymbol{\theta}_{m-1}(2)\times\Delta\boldsymbol{\theta}_{m-1}(3)+C\,\Delta\boldsymbol{\theta}_{m-1}(3)\times\Delta\boldsymbol{\theta}_m(1) \tag{25}$$

The type of optimization leading to algorithms E and F can be applied here also. The exact coning correction over a minor interval in a pure coning environment is, from Eq. (7),

$$\Delta\boldsymbol{\theta}_c=\frac{ab}{2}(\Omega\Delta T-\sin\Omega\Delta T)\boldsymbol{K} \tag{26}$$

Expanding into a series approximation yields the result

$$\Delta\boldsymbol{\theta}_c=ab\left[\frac{1}{2}\frac{\lambda^3}{6}-\frac{\lambda^5}{129}+\frac{\lambda^7}{5040}-\cdots\right]\boldsymbol{K} \tag{27}$$

The coning correction provided by the algorithm defined by Eq. (25) has exactly the same value as for algorithms E and F given by Eq. (15). Therefore, comparing the series expansions defined by Eqs. (21) and (27), it is clear that the following conditions must be satisfied simultaneously to optimize the algorithm:

$$2A+B+C=1/12$$

and

$$6A+B+C=1/60$$

which leads to the values $A=-1/60$, and $B+C=7/60$. Since the second condition imposes a constraint only on the sum of B and C, either B or C can be chosen as zero to simplify the algorithm. By making B equal to zero, algorithm G results.

Algorithm H

Consider the second special case that leads to algorithm H. The situation addressed here is one where two intervals are left over after applying either algorithm E or F to a set of subminor intervals that is some multiple of three. The coning correction over these two intervals is then computed using the sensor data over these remaining intervals, plus that occurring over the previous subminor interval. The structure of the algorithm is exactly the same as for algorithm E (with the differences noted), and is given by

$$\Delta\boldsymbol{\theta}_c = A\,\Delta\boldsymbol{\theta}_{m-1}(3)\times\Delta\boldsymbol{\theta}_m(2) + B\Delta\boldsymbol{\theta}_{m-1}(3)\times\Delta\boldsymbol{\theta}_m(1) + C\Delta\boldsymbol{\theta}_m(1)\times\Delta\boldsymbol{\theta}_m(2) \tag{28}$$

The optimal values of the coefficients A, B, and C are found in the manner previously established. The exact coning correction over the two subminor intervals in a pure coning environment is, from Eq. (7),

$$\Delta\boldsymbol{\theta}_c = \frac{ab}{2}(2\Omega\Delta T - \sin 2\Omega\Delta T)\boldsymbol{K} \tag{29}$$

Expanding into a series approximation yields the result

$$\Delta\boldsymbol{\theta}_c = \frac{ab}{2}\left[\frac{(2\lambda)^3}{6} - \frac{(2\lambda)^5}{120} + \frac{(2\lambda)^7}{5040} - \cdots\right]\boldsymbol{K} \tag{30}$$

The coning correction provided by the algorithm for this case is also exactly the same value as for algorithms E and F given by Eq. (15). Therefore, by comparing the series expansions defined by Eqs. (21) and (30), the conditions that must be satisfied simultaneously to optimize the algorithm are found to be

$$2A + B + C = 2/3$$

and

$$6A + B + C = 8/15$$

which leads to the values $A = -1/30$, and $B + C = 11/15$. As in algorithm G, the constraint applies only to the sum of B and C and, therefore, by choosing one or the other zero, the algorithm may be simplified. The choice $B = 0$ leads to algorithm H.

Algorithm Performance in a Generalized Environment

The optimization process leading to algorithms E, F, G, and H is predicated on the assumption of a pure coning environment. A justifiable concern is, therefore, that the optimality properties of the algorithms may fail to carry over to more general environments typically experienced in real applications. As it turns out, the optimization process that assumes a pure coning environment has validity in the general case as well. However, in demonstrating that this is true, it is necessary to establish a suitable criterion for optimality of an algorithm.

In the pure coning environment defined by Eq. (6), both the exact coning correction over some interval and the algorithmic approximation to it are constant and independent of the absolute time at which the interval occurred. The optimization of the algorithm is then based simply on minimizing the difference between the true and approximate coning corrections. In a more general environment, a suitable extension of the optimality criterion is that the difference between the mean values of the exact and approximate coning corrections should be minimized. This is an inherently good criterion, since the only real concern in generating the attitude reference solution is bias-like errors introduced either through the gyro outputs, or through the computational algorithms themselves. Small-amplitude zero-mean oscillatory errors are of little importance.

A generalized angular rate environment can be defined as a mixture of an infinite number of sinusoidal components according to

$$\boldsymbol{\omega} = \sum_{i=1}^{\infty}[a_i\Omega_i\cos(\Omega_i t + \phi_i)]\boldsymbol{I} + \sum_{i=1}^{\infty}[b_i\Omega_i\sin(\Omega_i t + \gamma_i)]\boldsymbol{J} \tag{31}$$

where ϕ_i and γ_i are the phase angles associated with a particular frequency Ω_i.

It is possible to repeat all of the steps leading to Eqs. (16–20) with the use of the angular rate vector defined by Eq. (31) in place of that defined by Eq. (6). If, in addition, only the mean values of the nominal and approximate coning corrections are retained (ignoring all other oscillatory components), it is possible after considerable algebraic and trigonometric combination and reduction to obtain the mean values of the exact and algorithmic corrections over the mth minor interval. The mean value of the exact correction in the third axis is given by

$$<\Delta\theta_c> = \sum_{i=1}^{\infty}\frac{a_i b_i}{2}(\Omega_i T - \sin\Omega_i T)\cos(\gamma_i - \phi_i) \tag{32}$$

and the mean value of the approximate correction provided by the algorithm given by

$$<\Delta\tilde{\theta}_c> = \sum_{i=1}^{\infty} a_i b_i\{A(2\sin 2\lambda_i - \sin\lambda_i - \sin 3\lambda_i) + (B + C)(2\sin\lambda_i - \sin 2\lambda_i)\}\cos(\gamma_i - \phi_i) \tag{33}$$

where

$$\lambda_i = \Omega_i\Delta T$$

The difference, $<\delta\theta_c>$, between the mean values of the two corrections is then

$$<\delta\theta_c> = \sum_{i=1}^{\infty} a_i b_i[(\Omega_i T - \sin\Omega_i T)/2 - A(2\sin 2\lambda_i - \sin\lambda_i - \sin 3\lambda_i) - (B + C)(2\sin\lambda_i - \sin 2\lambda_i)]\cos(\gamma_i - \phi_i) \tag{34}$$

in which the summand, exclusive of the cosine term, will be recognized as the absolute coning error in a pure coning environment at a frequency Ω_i. It is more convenient to rewrite Eq. (34) in terms of the per-unit errors in the algorithm at each frequency by multiplying numerator and denominator by the nominal coning correction defined by Eq. (7), which leads to the result

$$<\delta\theta_c> = \sum_{i=1}^{\infty}\frac{a_i b_i}{2}(\Omega_i T - \sin\Omega_i T)E_i\cos(\gamma_i - \phi_i) \tag{35}$$

where E_i is the per-unit algorithm error for pure coning at a frequency Ω_i, explicitly expressed by

$$E_i = 1 - \frac{1}{\tfrac{1}{2}(\Omega_i T - \sin\Omega_i T)}[A(2\sin 2\lambda_i - \sin\lambda_i - \sin 3\lambda_i) + (B + C)(2\sin\lambda_i - \sin 2\lambda_i)]$$

It is possible to show, that for the infinite mixture of sinusoids defined by Eq. (31), the mean value of the product of the angular displacement in one axis with the angular rate in the cross axis is given by

$$<\theta_1\omega_2> = \sum_{i=1}^{\infty}\frac{a_i b_i\Omega_i}{2}\cos(\gamma_i - \phi_i) \tag{36}$$

The copower of the angular displacement in one axis with the angular rate in the cross axis can be defined at a single frequency f_i in terms of the relationship

$$P_i = \frac{a_i b_i\Omega_i}{2}\cos(\gamma_i - \phi_i) = \Phi(f_i)\Delta f$$

where

P_i = copower at the ith frequency
Φ = cross-power spectral density function for θ_1,ω_2, in (rad^2/s)/Hz
Δf = frequency difference between successive frequencies, in Herz

In the limit, as the number of sinusoidals becomes infinite and the frequency difference becomes infinitesimally small, the relationship defined by Eq. (36) approaches the value

$$<\theta_1\omega_2> = \int_0^\infty \Phi(f)\,\mathrm{d}f \tag{37}$$

In this limiting situation, the algorithm error defined by Eq. (35) approaches the value

$$<\delta\theta_c> = \int_0^\infty \Phi(f)E(\omega)[(\omega T - \sin\omega T)/\omega]\,\mathrm{d}f \tag{38}$$

where $\omega = 2\pi f$, $E(\omega)$ denotes the per-unit algorithm error for pure coning at a frequency ω, and T is the duration of the minor interval.

It is clear from Eq. (38) that the selection of algorithm coefficients that make the mean algorithm error smallest for a given frequency will make it smallest over a spectrum of frequencies and, accordingly, the optimization procedure that leads to algorithms E, F, G, and H is valid in the general.

An alternate way of expressing the algorithm error in a vibrational environment is to divide the mean error in the coning correction over a minor computational interval, as defined by Eq. (38), by the duration of the minor interval T, which leads to the following expression for the error in the mean coning rate provided by the algorithm:

$$<\delta\omega_c> = \int_0^\infty \omega(f)E(\omega)[1 - \sin\omega T/(\omega T)]\,\mathrm{d}f \tag{39}$$

Although derived specifically for the class of algorithms represented by algorithm E, the result expressed by Eq. (39) is completely general, with the use of the applicable function $E(\omega)$.

Algorithm Performance in a Benign Environment

Suppose that the angular rate being experienced is smooth rather than oscillatory, as assumed in the previous discussion. The question arises of whether the optimization procedure that maximizes algorithm performance in the oscillatory environment compromises algorithm performance in a benign environment. To simplify the treatment of this question, assume that the actual angular rate history is such that, over any given minor interval, the angular rate polynomial given by Eq. (6) applies exactly. Then it is clear that, when such is the case, algorithm D will produce no error whatsoever. The errors associated with algorithm E, which has its coefficients adjusted to maximize its performance in an oscillatory environment, can be determined as follows.

For the class of algorithms to which algorithm E belongs, the correction provided by the algorithm over the mth minor interval is determined from Eqs. (15) and (13) as

$$\begin{aligned}\Delta\boldsymbol{\theta}_c &= A\,\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_3 + B\,\Delta\boldsymbol{\theta}_1\times\Delta\boldsymbol{\theta}_2 + C\,\Delta\boldsymbol{\theta}_2\times\Delta\boldsymbol{\theta}_3\\ &= A\left(2\boldsymbol{A}\times\boldsymbol{B}\Delta T^3 + 6\boldsymbol{A}\times\boldsymbol{C}\Delta T^4 + \frac{7}{3}\boldsymbol{B}\times\boldsymbol{C}\Delta T^5\right)\\ &+ B\left(\boldsymbol{A}\times\boldsymbol{B}\Delta T^3 + 2\boldsymbol{A}\times\boldsymbol{C}\Delta T^4 + \frac{2}{3}\boldsymbol{B}\times\boldsymbol{C}\Delta T^5\right)\\ &+ C\left(\boldsymbol{A}\times\boldsymbol{B}\Delta T^3 + 4\boldsymbol{A}\times\boldsymbol{C}\Delta T^4 + \frac{11}{3}\boldsymbol{B}\times\boldsymbol{C}\Delta T^5\right)\end{aligned}$$

Substituting for $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ from Eq. (14) and utilizing the values $A = 33/80$, $B = C = 57/80$ for algorithm D, and the values $A = 9/20$, $B = C = 27/40$ for algorithm E, results in the error associated with algorithm E, defined by

$$\Delta\boldsymbol{\theta}_c - \Delta\tilde{\boldsymbol{\theta}}_c = \frac{3}{40}(\boldsymbol{B}\times\boldsymbol{C})\Delta T^5 \tag{40}$$

It is clear that, by optimizing the algorithm for an oscillatory environment, the algorithm accuracy in a benign environment has been slightly compromised. However, for typical values of $\boldsymbol{B}$, $\boldsymbol{C}$, and ΔT, the given error should be negligible and, in any case, does not produce a sustained bias-like error, but an error that persists only for the duration of the assumed maneuver. The error in algorithm E′ is exactly the same as that given by Eq. (40) for algorithm E, since the two algorithms are algebraically equivalent to each other.

Repeating the preceding procedure for algorithm F, with $A = 9/20$, $B = 0$, and $C = 27/20$, shows that the comparable algorithm error is given by

$$\Delta\boldsymbol{\theta}_c - \Delta\tilde{\boldsymbol{\theta}}_c = -\frac{27}{20}(\boldsymbol{A}\times\boldsymbol{C})\Delta T^4 - \frac{39}{20}(\boldsymbol{B}\times\boldsymbol{C})\Delta T^5 \tag{41}$$

which clearly shows that this algorithm, while somewhat more efficient computationally than algorithm E′, produces a larger error in the benign environment. However, as noted, the nature of this type of error is such as to be of minimal concern, the greater concern being with the bias-like algorithm errors that are produced in a sustained vibrational environment.

Conclusion

The paper has examined the properties of various algorithms for integrating the noncommutivity rate equation, and has defined the errors associated with each in a pure coning environment – the clear superiority in this regard of the class of optimized algorithms derived in the paper being noteworthy. The optimization process associated with this class of algorithms was shown to also maximize performance in a generalized vibrational environment. An expression defining the mean coning rate error in such environments was derived, and provides an effective means of evaluating algorithm performance when the system is undergoing random vibratory angular motions. The paper also showed that the optimization procedure that maximizes performance in a pure coning environment results in a slight compromise of algorithm performance in more benign environments, but this is usually a consideration of secondary importance.

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References

[1]Bortz, J., "A New Mathematical Formulation for Strapdown Inertial Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES 7, No. 1, Jan. 1973.

[2]Jordan, J., "An Accurate Strapdown Direction Cosine Algorithm," NASA TN-D-5384, Sept. 1969.

[3]Miller, R., "A New Strapdown Attitude Algorithm," *Journal of Guidance, Control, and Dynamics*, Vol. 6, July-Aug. 1983.

[4]McKern, R., "A Study of Transformation Algorithms for Use in a Digital Computer," M. S. Thesis, Massachusetts Inst. of Technology, Cambridge MA, 1968.

[5]Edwards, A., "The State of Strapdown Inertial Guidance and Navigation," *Navigation*, Vol. 18, No. 4, Winter 1971–72.